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ON THE STRAIN ENERGY OF LAMINATED COMPOSITE PLATES†

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Abstract—A strain energy function is obtained for nonhomogeneous, laminated, composite plates for the case when each lamina exhibits monoclinic material symmetry about its middle surface. The starting point is the three-dimensional strain energy based on geometrically nonlinear elasticity theory. The variational-asymptotical method is used to decompose the nonlinear three-dimensional problem into two separate problems: (1) a linear, through-the-thickness, one-dimensional analysis to obtain appropriate plate elastic constants and relations between plate deformation variables and three-dimensional results; and (2) a nonlinear, two-dimensional analysis to analyse the plate deformation. Closed-form analytical expressions are derived for the plate elastic constants as well as the displacement and strain distributions through the thickness of the plate. Even with this generality, no more variables are involved than in Reissner–Mindlin plate theory. Also, in spite of the simple form for the plate strain energy, there are no restrictions on the magnitudes of displacement and rotation measures. Two approximations are obtained in the through-the-thickness analysis, the first being equivalent to classical laminated plate theory, and the second incorporating shear deformation effects. The first approximation is asymptotically correct for plates of the form considered. The second approximation is asymptotically correct for plates with certain additional material restrictions. In applying the method, one first solves the through-the-thickness problem and then uses the resulting elastic constants to pose the nonlinear plate problem. After solving the nonlinear problem, one substitutes these results back into the linear three-dimensional relations for displacement and strain throughout the plate.

INTRODUCTION

When a flexible structure has one dimension that is much smaller than the other two, it can often be treated as a plate, a two-dimensional structure. Many engineering structures can be idealized as plates, leading to much simpler equations than would be obtained if three-dimensional elasticity were used to model them.

Although dimensional reduction processes can be simple for homogeneous, isotropic plates, and especially for restricted cases of deformation, they are far less tractable for nonhomogeneous, laminated composite plates undergoing arbitrary deformation. Specifically, difficulties arise in obtaining a two-dimensional strain energy function that is equivalent, at least in some sense, to the three-dimensional representation. For nonhomogeneous, anisotropic plates, all possible deformations of the three-dimensional structure must be included in the formulation. This in turn suggests that it is necessary to remove the well-known restrictions that are typically imposed in a plate analysis.

Kirchhoff's classical plate analysis reveals the well-known behavior for linear deformation of homogeneous, isotropic plates [see Reissner (1985) and the references cited therein]. A line element of material normal to the plate mid-surface when the plate is undeformed remains straight and normal to the reference surface of the deformed plate during pure extension or pure bending deformation. The length of such line elements is contracted ("out-of-plane" warping) due to Poisson effects which are manifested in the transverse normal strain. This behavior allows the determination of six different stiffnesses. One extensional stiffness (the same in two directions), one stiffness coupling the extension in two orthogonal directions due to Poisson effects, one bending stiffness (the same in two

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directions), one stiffness coupling the bending in two orthogonal directions due to Poisson effects, one in-plane shear stiffness, and one twisting stiffness.

In order to relax the requirement that originally normal line elements remain normal, one must introduce into the displacement field two additional variables which depend on the surface coordinates and govern the orientation of the originally normal line element. In addition, points aligned along an originally normal line element do not remain on a line normal to the deformed plate ("in-plane" warping). The Reissner–Mindlin shear deformation theory incorporates one additional stiffness (the same in two directions) due to transverse shear. The shear stiffness can also be written in terms of the shear modulus of the material, the thickness of the plate, and a factor of $\frac{5}{6}$, the so-called "shear correction" factor. The transverse shear stress turns out to be of a higher order than the in-plane stresses by a factor of h/l where h is the thickness of the plate, and l is the wavelength of the deformation pattern.

Such simple characterizations are not possible for analysis of nonhomogeneous, anisotropic plates. There are some simplifications for laminated plates, if by "laminated" we mean plates constructed of individual thin plates, each one of which having monoclinic material symmetry about its midplane. Even this subset of the more general problem is a monumental challenge and has attracted an enormous amount of attention. Due to space limitations it is not possible to cite more than a fraction of the work in this field—even in the last decade. The reader is encouraged to refer to the recent review papers of Noor and Burton (1989) and Librescu and Reddy (1989) on laminated composite plate modeling. These, along with the comprehensive discussion on the history and the development of plate theories by Reissner (1985) should give the interested reader an appreciation for the difficulties involved in this subject.

In much of the literature cited in these review papers, it is typical for theories to have more displacement/rotation variables than the Reissner–Mindlin theory. This is done in two ways: (1) displacement variables which are associated with the individual layers can be carried along, the number of which depends on the number of layers in the plate; (2) the displacement field may be expanded through the thickness in terms of polynomials, and the higher-order coefficients can then be carried along as additional variables. The former approach involves a large number of degrees of freedom in finite element implementations. The latter, because of possible discontinuities in material properties between lamina, is incapable of giving exact reproduction of quantities which have discontinuous values or derivatives through the thickness. Thus this method may not yield sufficiently accurate results for displacement, strain and stress distributions throughout the plate, and it may not suffice to accurately determine elastic constants.

Even if the number of variables for an anisotropic plate theory were constrained to be the same as that of the Reissner–Mindlin theory, there could be elastic couplings among all the global deformations. This means that instead of seven fundamentally different stiffnesses, there could be as many as 36 (a fully populated, symmetric 8×8 matrix). The in- and out-of-plane warping deformations may be coupled. If the plate is restricted to be laminated as defined above, then certain of the 36 constants will vanish, and the calculation of others will be simplified. In order to obtain the correct elastic couplings in a consistent manner, all possible deformations of the plate would need to be included.

Therefore, in this paper our starting point is a general three-dimensional analysis of deformation. This general representation will then be simplified by use of the variational-asymptotical method, developed by Berdichevsky (1979). The present work extends the work of Berdichevsky (1979) to yield an approximation of the strain energy for nonhomogeneous, laminated plates. Finally, relations are derived which allow for complete recovery of the three-dimensional displacement and strain (or stress) fields from the results of a two-dimensional analysis. In spite of this generality, the number of variables remains the same as in Reissner–Mindlin plate theory.

THREE-DIMENSIONAL FORMULATION

A plate is a flexible body in which matter is distributed about a planar surface so that one dimension, h, is significantly smaller than the other two. (Although much of the present analysis can be easily extended to treat shells, herein we consider only plates.) The plate is assumed to be nonhomogeneous only through the thickness, consisting of possibly distinct, possibly layered, anisotropic materials; note that material properties and their distribution vary *only* through the thickness of the plate. Furthermore, we assume that the plate is constructed of individual thin plates, or laminae, each one of which has monoclinic material symmetry about its mid-plane. This way, there is no elastic coupling in the three-dimensional strain energy between transverse normal and transverse shear strains or between in-plane and transverse shear strains. This still allows the present formulation to be applicable to laminated, fiber-reinforced composite plates which are typical in current engineering practice, because the fiber directions in such undeformed plates are parallel to the plane of the undeformed plate.

In this section, the three-dimensional strain field is developed first, giving emphasis on three-dimensional plate geometry. The three-dimensional elastic matrix, which relates the strain to its conjugate stress given next. The strain energy will then be developed and decomposed into two positive definite quadratic forms, with which more physical understanding of the problem is gained.

Development of the strain field

Now we turn to the three-dimensional strain field. Throughout the analysis, Greek indices assume values 1 or 2, Latin indices assume values 1, 2 and 3 and repeated indices are summed over their ranges.

Undeformed plate geometry. Let us establish a Cartesian coordinate system x_i so that x_α denote lengths along orthogonal lines in the mid-surface and $x_3 = h\zeta$ is the distance of an arbitrary point to the geometric mid-surface in the undeformed plate. Here ζ is the non-dimensional thickness coordinate, and the thickness h is taken to be a constant.

Let $\mathbf{b}_i$ denote an orthogonal reference triad along the coordinate lines of the undeformed plate. The position vector from a fixed point O to an arbitrary point is

$$\hat{\mathbf{r}}(x_1, x_2, \zeta) = \mathbf{r}^*(x, x_2) + h\zeta\mathbf{b}_3 = x_\alpha\mathbf{b}_\alpha + h\zeta\mathbf{b}_3. \tag{1}$$

Here $\mathbf{r}^*$ is the position vector from O to an arbitrary point on the mid-surface of the undeformed plate. Thus

$$\langle\hat{\mathbf{r}}(x_1, x_2, \zeta)\rangle \triangleq \int_{-1/2}^{1/2} \hat{\mathbf{r}}(x_1, x_2, \zeta)\,\mathrm{d}\zeta \triangleq \mathbf{r}^*(x_1, x_2), \tag{2}$$

where the use of angled brackets to denote the above integral will be used throughout the rest of the development.

Deformed plate geometry. In a similar manner, consider the deformed plate configuration. The particle which had position vector $\hat{\mathbf{r}}(x_1, x_2, \zeta)$ in the undeformed plate now has position vector $\hat{\mathbf{R}}(x_1, x_2, \zeta)$. The specific form of $\hat{\mathbf{R}}$ must await the introduction of several entities related to the deformation.

To this end, we introduce another orthonormal triad $\mathbf{B}_i(x_1, x_2)$, called the deformed plate triad. The orientation of $\mathbf{B}_i$ relative to $\mathbf{b}_i$ can be specified by an arbitrarily large rotation, and $\mathbf{B}_i$ is coincident with $\mathbf{b}_i$ when the plate is undeformed. Rotation from $\mathbf{b}_i$ to $\mathbf{B}_i$ is described in terms of a matrix of direction cosines $C(x_1, x_2)$ such that

$$\mathbf{B}_i = C_{ij}\mathbf{b}_j, \quad C_{ij} = \mathbf{B}_i \cdot \mathbf{b}_j. \tag{3}$$

Once a specific form of the displacement field is introduced, the matrix χ whose elements are defined by

$$\chi_{ij} = \mathbf{B}_i \cdot \frac{\partial \hat{\mathbf{R}}}{\partial x_j}, \tag{4}$$

can be found. Now, following Danielson and Hodges (1987), the polar decomposition theorem shows that χ can be uniquely decomposed into an orthogonal rotation matrix $\hat{C}$ times a symmetric right stretch matrix U,

$$\chi = \hat{C}U. \tag{5}$$

Note that the elements of the matrix $\hat{C}$ are components of the usual tensor of rotation resulting from the polar decomposition theorem post-multiplied by C^T, because χ is resolved in mixed bases as shown in eqn (4). The matrix of Jaumann strain components is then defined by

$$\hat{\Gamma} = U - I_3, \tag{6}$$

where I_3 is the 3×3 identity matrix and $\hat{\Gamma}$ is a 3×3 symmetric matrix containing the Jaumann strain components. In accordance with Danielson and Hodges (1987), the rotation represented by $\hat{C}$ may be specified by a finite rotation vector $\Phi_i(x_1, x_2, x_3)\mathbf{B}_i(x_1, x_2)$ so that

$$\hat{C} = e^{\tilde{\Phi}}, \tag{7}$$

where $\tilde{\Phi}$ denotes the antisymmetric matrix whose components are $\tilde{\Phi}_{12} = -\Phi_3$, $\tilde{\Phi}_{13} = \Phi_2$, $\tilde{\Phi}_{23} = -\Phi_1$. From eqns (5)–(7), we obtain the exact expression

$$\hat{\Gamma} = e^{-\tilde{\Phi}}\chi - I_3. \tag{8}$$

Using the estimation procedures of Berdichevsky (1979), it is possible to show that the elements of $\hat{\Gamma}$ and $\tilde{\Phi}$ are small quantities of the order of ε, where ε is the maximum value of the strain in the plate. This leads to the "small local rotation" theory of Danielson and Hodges (1987). Following Danielson (1991), we then retain only the lowest order terms in the Taylor series expansions of the matrices in eqn (8), and enforce the symmetry of the right-hand side, thus obtaining

$$\hat{\Gamma} = e^{-\tilde{\Phi}}\chi - I_3, \quad \approx (I - \tilde{\Phi})[I + (\chi - I)] - I, \quad \approx \chi - I - \tilde{\Phi} = \chi^T - I + \tilde{\Phi}. \tag{9}$$

The following expressions for $\tilde{\Phi}$ and Γ can be obtained as

$$\tilde{\Phi} = \tfrac{1}{2}(\chi - \chi^T), \quad \hat{\Gamma} = \tfrac{1}{2}(\chi + \chi^T) - I. \tag{10}$$

Here $\hat{\Gamma}$ is a 3×3 symmetric matrix containing the Jaumann strain components. The expression for $\hat{\Gamma}$ is then quite simple once the components of the deformation gradient are known. Since the local rotation is of the same order as the strain, it does not appear in the strain expression.

Specification of displacement field. Now, for the purpose of later obtaining the strain field in terms of generalized (i.e. surface) strain measures, we introduce the position vector from O to the points of the average surface of the deformed plate as

$$\langle \hat{\mathbf{R}}(x_1, x_2, \zeta) \rangle \triangleq \mathbf{R}^*(x_1, x_2) = \mathbf{r}^*(x_1, x_2) + \mathbf{u}^*(x_1, x_2), \tag{11}$$

where $\mathbf{u}^*$ (x_1, x_2) is a "displacement" vector, of sorts. This vector is properly understood as the position vector from a point (x_1^*, x_2^*) on the mid-surface of the undeformed plate to a point (x_1^*, x_2^*) on the average surface of the deformed plate. It should be evident that $\mathbf{u}^*$

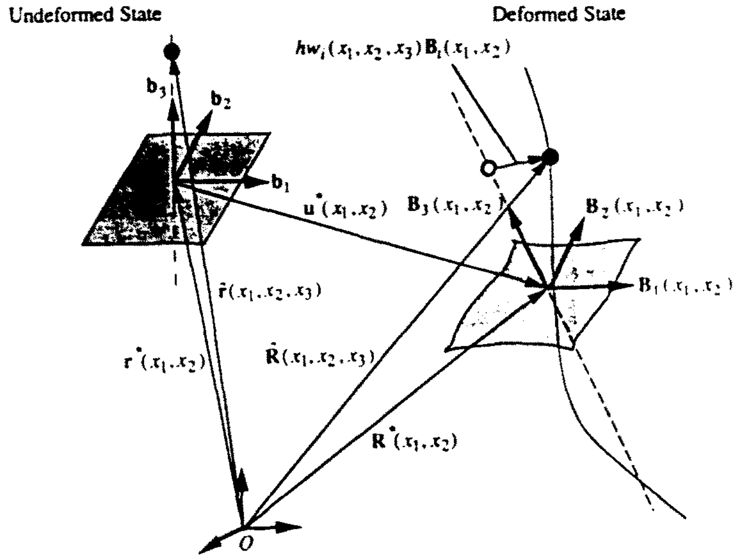

Fig. 1. Schematic of plate deformation.

is *not* the displacement of a particular material point on the mid-surface of the undeformed plate, which would be given by

$$(\hat{\mathbf{R}}-\hat{\mathbf{r}})|_{\zeta=0} = \mathbf{u}^*(x_1,x_2)+hw_i(x_1,x_2,0)\mathbf{B}_i(x_1,x_2). \tag{12}$$

Now the position vector from O to any arbitrary point on the plate can be written as

$$\hat{\mathbf{R}}(x_1,x_2,\zeta) = \mathbf{R}^*(x_1,x_2)+h\zeta\mathbf{B}_3(x_1,x_2)+hw_i(x_1,x_2,\zeta)\mathbf{B}_i(x_1,x_2), \tag{13}$$

where w_i is the warping of the normal line element. The resulting displacement field is shown schematically in Fig. 1. The shaded surfaces are infinitesimal elements of the average surfaces of the undeformed and deformed states. An arbitrary material point off the average surface is represented as a heavy dot. Note that eqn (13) includes all possible deformations. However, since the warping is a three-dimensional displacement, the function $\hat{\mathbf{R}}$ is six times redundant. This redundancy can be eliminated by imposing six conditions on the definitions of $\mathbf{R}^*$ and/or $\mathbf{B}_i$. In light of eqn (11) which defines $\mathbf{R}^*$ to be the average position through the thickness, one can easily show that the warping must satisfy the three constrains

$$\langle w_i(x_1,x_2,\zeta)\rangle = 0. \tag{14}$$

Equation (13) is still indeterminate until three more dependency relations are specified for $\mathbf{B}_i$ and/or $\mathbf{R}^*$. The relations can result in additional constraints on the warping, depending on the level of the approximation. One will be introduced immediately below, in the context of generalized strains; the other two must be dealt with later in the context of the reduction to two dimensions. These concern the orientation of the vector $\mathbf{B}_3$, defined below to depend on the level of accuracy to which the asymptotical approximation is taken. In the first approximation $\mathbf{B}_3$ is taken to be normal to the average surface of the deformed plate, whereas in the second approximation we take it to be associated with an average rotation of the normal line element.

Generalized strains. Danielson (1991) shows that it is possible to express the three-dimensional strain field in terms of two-dimensional quantities, the so-called generalized strain measures. These measures are given by

$$\varepsilon_{\alpha\beta} = \mathbf{R}^*_{,\alpha}\cdot\mathbf{B}_\beta - \delta_{\alpha\beta}, \quad K_{\alpha\beta} = -\mathbf{B}_{\beta,\alpha}\cdot\mathbf{B}_3, \quad 2\gamma^*_{\alpha 3} = \mathbf{R}^*_{,\alpha}\cdot\mathbf{B}_3, \tag{15}$$

where $(\ )_{,\alpha}$ denotes the partial derivative with respect to x_α. The vectors $\mathbf{B}_\alpha$ are taken as normal to $\mathbf{B}_3$ and can always be chosen so that $\varepsilon_{12} = \varepsilon_{21}$. (This is one of the three remaining dependency relations mentioned above. The others are introduced in the context of the dimensional reduction.) From the form of eqn (15), one should understand that adopting a different definition for $\mathbf{B}_3$ changes the expressions for the generalized strain measures in terms of displacement and rotational variables. The generalized strain measures can be arranged in matrix form for convenience as

$$\varepsilon = [\varepsilon_{11}\ 2\varepsilon_{12}\ \ \varepsilon_{22}]^{\mathrm{T}}, \quad K = [K_{11}\ 2\kappa_{12}\ K_{22}]^{\mathrm{T}}, \quad 2\gamma^* = [2\gamma^*_{13}\ 2\gamma^*_{23}]^{\mathrm{T}}, \tag{16}$$

where $2\kappa_{12} = K_{12}+K_{21}$ and ε and K account for extensional/in-plane shear and bending/twisting, respectively; and $2\gamma^*_{\alpha 3}$ accounts for transverse shear strain. In the first approximation, $2\gamma^*_{\alpha 3} = 0$; whereas this is not the case in the second approximation.

Deformation gradient. With these definitions for the generalized strains one can now derive the matrix χ from which the strain field can be obtained directly. Substitution of eqn (13) into eqn (4), making use of eqns (16), one obtains

$$\chi = \begin{bmatrix} 1+\varepsilon_{11}+h\zeta K_{11}+hw_{1,1} & \varepsilon_{12}+h\zeta K_{21}+hw_{1,2} & w'_1 \\ \varepsilon_{12}+h\zeta K_{12}+hw_{2,1} & 1+\varepsilon_{22}+h\zeta K_{22}+hw_{2,2} & w'_2 \\ 2\gamma^*_{13}+hw_{3,1} & 2\gamma^*_{23}+hw_{3,2} & 1+w'_3 \end{bmatrix},$$

where the only nonlinear terms, which are products of $K_{\alpha i}$ and hw_i, have been neglected based on the small strain hypothesis. Now, from the second of eqn (10) the Jaumann strain field is determined.

Strain energy

One can group the Jaumann strain into three different column matrices from physical considerations as follows

$$\Gamma_e = [\Gamma_{11}\ 2\Gamma_{12}\ \Gamma_{22}]^{\mathrm{T}}, \quad 2\Gamma_s = [2\Gamma_{13}\ 2\Gamma_{23}]^{\mathrm{T}}, \quad \Gamma_t = \Gamma_{33}, \tag{17}$$

where Γ_e includes the extensional and in-plane shearing strains, and Γ_s and Γ_t contain the transverse shear and transverse normal strains, respectively. Thus, the Jaumann strain can now be written as

$$\Gamma = [\Gamma_e\ 2\Gamma_s\ \Gamma_t]^{\mathrm{T}}. \tag{18}$$

A similar procedure can be followed for the three-dimensional Jaumann stress, which is conjugate to the Jaumann strain, as shown by Ogden (1984), so that

$$Z_e = [Z_{11}\ Z_{12}\ Z_{22}]^{\mathrm{T}}, \quad Z_s = [Z_{13}\ Z_{23}]^{\mathrm{T}}, \quad Z_t = Z_{33}, \tag{19}$$

where Z_e is comprised of the extensional and in-plane shear stresses while Z_s and Z_t contain the transverse shear and transverse normal stresses. Therefore, the Jaumann stress may be rewritten as

$$Z = [Z_e\ Z_s\ Z_t]^{\mathrm{T}}. \tag{20}$$

In light of the above development, the three-dimensional constitutive law between the Jaumann stress and its conjugate Jaumann strain can be expressed as

$$\begin{Bmatrix} Z_e \\ Z_s \\ Z_t \end{Bmatrix} = \begin{bmatrix} D_e & D_{es} & D_{et} \\ D_{es}^T & D_s & D_{st} \\ D_{et}^T & D_{st}^T & D_t \end{bmatrix} \begin{Bmatrix} \Gamma_e \\ 2\Gamma_s \\ \Gamma_t \end{Bmatrix}, \tag{21}$$

where D_e, D_{es}, D_{et}, D_s, D_{st} and D_t are 3×3, 3×2, 3×1, 2×2, 2×1 and 1×1 matrices, respectively. This material elastic law is written with respect to the plate Cartesian axes, which are not necessarily along any particular material fiber direction because of the nonhomogeneity of the plate. In other words, the material constants used in eqn (21) must be the transformed values, from whatever local coordinate systems may be natural to the individual laminae, to the Cartesian coordinate system of the plate, x_i.

Recalling that h is constant over the entire plate, the strain energy per unit area of the plate can then be written as

$$hJ = \frac{h}{2}\langle Z^T\Gamma\rangle. \tag{22}$$

Let us now decompose the strain energy into two positive definite, quadratic forms. Here we define the extensional strain energy $J_{\parallel}$, and the transverse strain energy $J_{\perp}$ (containing contributions from both transverse normal and shear strains) as

$$J_{\parallel} = \min_{\Gamma_s,\Gamma_t} J, \quad J_{\perp} = J - J_{\parallel}. \tag{23}$$

Following Berdichevsky (1979), it can be shown that this representation is unique.

Up to this point in the development, the material properties are still completely general. In order to proceed analytically, we found it helpful to simplify the formulation by specializing eqn (21) somewhat. When each lamina exhibits a monoclinic symmetry (e.g. when fibers in the undeformed plate are oriented parallel to the plane of the undeformed plate), the 3×2 matrix D_{es} and the 2×1 matrix D_{st} both vanish. In this case, the extensional and transverse energies can be written in terms of the three-dimensional material properties as follows:

$$2J_{\parallel} = \langle \Gamma_e^T D_{\parallel}\Gamma_e\rangle, \quad 2J_{\perp} = \langle 2\Gamma_s^T D_s 2\Gamma_s + D_t(\Gamma_t + D_{\perp}\Gamma_e)^2\rangle, \tag{24}$$

where

$$D_{\parallel} = D_e - D_{et}D_t^{-1}D_{et}^T, \quad D_{\perp} = D_t^{-1}D_{et}^T. \tag{25}$$

After completing the preliminaries of the three-dimensional strain, stress and the strain energy, we can now turn our attention to the two-dimensional plate modeling.

DIMENSIONAL REDUCTION

In plate formulations, we attempt to do the impossible—that is, to reproduce, in a two-dimensional body, the energy stored in a three-dimensional one. This process *cannot be performed in an exact manner*. However, due to the interest of working with simpler systems with smaller dimensions, researchers have turned to asymptotical methods in order to reduce the dimension of the model for bodies which contain one or more small parameters. Plates are such bodies, because the thickness of the plate is much smaller than the other two dimensions.

Thus, in what follows we replace the three-dimensional plate problem by an *approximate* two-dimensional problem in which the strain energy will only be a function of the surface coordinates. This will be done with the aid of the variational–asymptotical formulation originally developed by Berdichevsky (1979). Before getting into the application of this method, we give a brief overview of it. Then we will develop the first and second approximations for the laminated plate problem.

Variational–asymptotical method

This formulation is derived for functionals with small parameters, unlike usual asymptotical formulations for differential equations with small parameters. Rather than substituting into the functional all the asymptotical orders of the unknown functions at the beginning, order assessment is done after each approximation.

Consider a given three-dimensional function $\mathscr{F}(\Gamma, h)$ with a small parameter h. We decompose this functional into two parts

$$\mathscr{F}(\Gamma,h) = \mathscr{E}_1(\Psi,z_1)+\mathscr{E}_h(\Psi,z_1,h), \tag{26}$$

such that $\mathscr{E}_1$ is obtained by discarding all smaller contributions to the energy, which are represented by $\mathscr{E}_h$. Here Ψ is a function of surface coordinates only (in our case these correspond to the generalized strains) and z_1 is a function of all three coordinates (in our case it corresponds to the warping). Let us minimize the functional, first considering only the main contribution $\mathscr{E}_1$

$$\min_{z_1} \mathscr{F} = \min_{z_1} \mathscr{E}_1(\Psi,z_1) \triangleq \mathscr{F}_1. \tag{27}$$

Solution of the Euler equations of this functional can be written symbolically as

$$z_1 = y_1(\Psi,\zeta), \tag{28}$$

which will be designated as the first minimizing function, only if the second of such functions can be shown to be of higher order. Therefore, there is a need to develop the second aproximation. To do so, we introduce a minimizing function for the second approximation so that

$$z_1 = z_2+y_1(\Psi,\zeta). \tag{29}$$

Substituting this into the original functional eqn (26), one will obtain

$$\mathscr{F}(\Gamma,h) = \mathscr{F}_1(\Psi)+\mathscr{E}_2(\Psi,z_2,h)+\mathscr{E}_{h1}(\Psi,z_2,h), \tag{30}$$

where $\mathscr{E}_2$ is obtained by discarding all the other smaller contributions, which are now represented by $\mathscr{E}_{h1}$. Then the minimum of the functional becomes

$$\min_{z_2} \mathscr{F} = \min_{z_2} \mathscr{E}_2(\Psi,z_2,h) \triangleq \mathscr{F}_2, \tag{31}$$

from which the second minimizing function can be obtained as

$$z_2 = y_2(\Psi,\zeta). \tag{32}$$

If this is of higher order than z_1, then z_1 and $\mathscr{F}_1$ constitute the *first approximation*. If this is not the case, then the estimations for the first and the second approximations must be corrected and the procedure repeated.

Obviously, in order to prove that z_2 is the second approximation, the third approximation must be developed in a similar manner. The sequence of approximations may be stopped whenever it is desired. The original functional can then be written after the kth approximation as

$$\mathscr{F}(\Gamma, h) = \mathscr{F}_1(\Psi)+\mathscr{F}_2(\Psi)+\cdots+\mathscr{E}_k(\Psi)+\mathscr{E}_{k+1}(\Psi, z_{k+1}, h)+\mathscr{E}_{hk}(\Psi, z_{k+1}, h) \quad (33)$$

where $z_{k+1} = z_k - y_k(\Psi, \zeta)$. It can be seen that the two-dimensional functional is now asymptotically equivalent to the original three-dimensional one.

In the following sections, we will apply this method for nonhomogeneous, laminated plates in pursuit of the asymptotically correct strain energy. Before doing so, however, it is appropriate to discuss the estimation procedure. First, we introduce upper bounds on the in-plane and bending strain measures ϵ_e and ϵ_b such that

$$\sqrt{\epsilon^T\epsilon} \leqslant \epsilon_e, \quad \frac{h}{2}\sqrt{K^TK} \leqslant \epsilon_b. \quad (34)$$

Now, for the first approximation we choose to keep terms in the strain field that are of the order of ϵ where

$$(\epsilon_e+\epsilon_b) \leqslant \epsilon. \quad (35)$$

This implies that we will have strain energy density of the order $\mu\epsilon^2$ where μ is of the order of the elastic moduli.

Rather than write out complete expressions for the strain field, we will only write the terms needed to the appropriate order. The terms not written for the kth approximation contribute terms to the strain energy of the order $\mu\epsilon^2\,(h/l)^{2k-1}$ where $k \geqslant 1$ and where l is the smallest constant for which both of the following hold for all possible combinations of α and β

$$\sqrt{\epsilon_{,\alpha}^T\epsilon_{,\beta}} \leqslant \frac{\epsilon_e}{l}, \quad \frac{h}{2}\sqrt{K_{,\alpha}^TK_{,\beta}} \leqslant \frac{\epsilon_b}{l}. \quad (36)$$

First approximation

Following Berdichevsky (1979), we stipulate that $\mathbf{B}_3 = \mathbf{N}$ where $\mathbf{N}$ is the unit normal to the deformed plate average surface. It should be noted that $\mathbf{N}\cdot\mathbf{R}_{,\alpha}^* = 0$ and hence that $2\gamma_{\alpha 3}^* = 0$. These constitute the two remaining dependency relations in the context of the first approximation; they will not be used in the second approximation. Here we denote $\mathbf{B}_i$ as the deformed plate triad for the *first approximation*, whose orientation relative to $\mathbf{b}_i$ can be specified by an arbitrarily large rotation. For the first approximation only, the measure numbers of the rotation vector can be found in terms of the gradient of the displacements, in accord with the well-known Kirchhoff hypothesis.

We now need the terms in the Jaumann strain that contribute to the first approximation. To get these, we substitute eqn (16) into the second of eqns (10) and group the resulting terms in accordance with eqns (17). Finally, retaining only those terms that are of order ε, we obtain

$$\Gamma_e = \varepsilon+\zeta Kh, \quad 2\Gamma_s = w_\parallel', \quad \Gamma_t = w_3', \qquad \text{where} \quad w_\parallel = [w_1 \quad w_2]^T, \quad (37, 38)$$

and $(\ )'$ denotes differentiation with respect to ζ.

The extensional energy does not include any contribution from warping; thus, only the transverse energy needs to be minimized with respect to warping. Then, whether the minimized function is consistent with the above estimates will be explored. Now, the variational problem is written as follows:

$$2J_\perp = \langle w_\parallel'^T D_s w_\parallel' \rangle + \langle D_t[w_3' + D_\perp(\varepsilon+\zeta Kh)]^2 \rangle \quad (39)$$

to be minimized with the constraints from eqn (14).

$$\langle w_{\parallel} \rangle = \langle w_3 \rangle = 0. \tag{40}$$

It can be seen that the first part of the transverse energy, which is due to in-plane warping, is decoupled from the out-of-plane warping. The minimum value of the first term can be reached when $w_{\parallel} = 0$ with the constraints $\langle w_{\parallel} \rangle = 0$. Let us investigate the remaining part. The functional is now

$$2J_{\perp} = \langle D_t[w_3' + D_{\perp}(\varepsilon + \zeta K h)]^2 \rangle \tag{41}$$

with the constraint $\langle w_3 \rangle = 0$. First we define some indefinite integrals associated with the possibly discontinuous quantity $D_{\perp}$, so that

$$D_{\perp 1}' = -D_{\perp}, \quad D_{\perp 2}' = -\zeta D_{\perp}. \tag{42}$$

Note that inter-lamina continuity of $D_{\perp\alpha}$ must be maintained, leaving only one 1×3 matrix of integration constants for each of the eqns (42). Next we make a change of variable as

$$w_3 = D_{\perp 1}\varepsilon + D_{\perp 2}Kh + \overline{w_3}. \tag{43}$$

Then, the functional reduces to the simple form

$$2J_{\perp} = \langle D_t \overline{w_3}'^2 \rangle. \tag{44}$$

We need to find the transformed constraint for this functional. Applying the original constraint to eqn (43), one can obtain

$$\langle \overline{w_3} \rangle = 0, \tag{45}$$

if the integration constants from eqn (42) are chosen so that

$$\langle D_{\perp 1} \rangle = \langle D_{\perp 2} \rangle = 0. \tag{46}$$

Inter-lamina continuity must also be maintained on the warping displacement w_3, but discontinuities in the derivatives of w_3 are, of course, permitted.

Now to deal with the transformed variational problem is trivial, since the minimum of $J_{\perp}(\overline{w_3})$, with the constraint given by eqn (45), is reached when $\overline{w_3} = 0$. Thus, the warping is obtained for the first approximation of laminated plates as

$$w_{\parallel} = 0, \quad w_3 = D_{\perp 1}\varepsilon + D_{\perp 2}\mathrm{K}h \quad \text{(nonhomogeneous)}$$
$$w_3 = -\zeta D_{\perp}\varepsilon - \frac{1}{2}\left(\zeta^2 - \frac{1}{12}\right) D_{\perp} K h \quad \text{(homogeneous)}. \tag{47}$$

From this equation we see that the warping is of order ε. This is, in fact, consistent with our estimations. However, to accept that this is the first approximation, one needs to check the second approximation and confirm that the warping function of the second approximation is of a higher order than that of the first one. It will be seen in the next section that this is the case. The nature of the warping is simple: the in-plane warping is zero and the out-of-plane warping consists of the normal line element contracting or stretching in response to deformation involving ε and K.

Substituting the warping functions into the transverse energy, one can see that *in the first approximation the transverse energy is zero*. The total strain energy per unit area is then comprised only of the extensional energy

$$hJ = hJ_{\parallel} = \frac{h}{2}\langle(\varepsilon+\zeta Kh)^{\mathrm{T}} D_{\parallel}(\varepsilon+\zeta Kh)\rangle. \tag{48}$$

It is possible to define the force and moment stress resultants N and M, respectively, as follows:

$$N = \langle Z_e\rangle h = h\left(\frac{\partial J}{\partial \varepsilon}\right)^{\mathrm{T}}, \quad M = \langle \zeta Z_e\rangle h^2 = h\left(\frac{\partial J}{\partial K}\right)^{\mathrm{T}}, \tag{49}$$

which in turn yields the stiffness matrix given by classical laminated plate theory

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B^{\mathrm{T}} & D \end{bmatrix} \begin{Bmatrix} \varepsilon \\ K \end{Bmatrix}. \tag{50}$$

where

$$A = h\langle D_{\parallel}\rangle, \quad B = h^2\langle\zeta D_{\parallel}\rangle, \quad D = h^3\langle\zeta^2 D_{\parallel}\rangle, \tag{51}$$

where $D_{\parallel} = \bar{Q}$ the well-known transformed reduced stiffness matrix [see Jones (1975)].

The characteristics of the first approximation may be summarized as: (1) normal line elements of the undeformed plate remain straight and normal to the deformed plate average surface; (2) the transverse normal strain is *not* zero; (3) the transverse normal stress is zero; (4) the transverse shear stresses are zero; (5) the transverse energy is zero; and (6) for both laminated and isotropic cases, the first approximation coincides with *classical* plate theories.

Second approximation

In the context of the second approximation, $\mathbf{B}_3(x_1, x_2)$ is not necessarily parallel to $\mathbf{N}$. Thus, in addition to the three constraints in eqn (14) and the choice of $\mathbf{B}_\alpha$ such that $\varepsilon_{12} = \varepsilon_{21}$, the warping can now be shown to satisfy the two additional constraints

$$\langle\zeta w_\alpha(x_1, x_2, \zeta)\rangle = 0 \tag{52}$$

which also fixes the orientation of $\mathbf{B}_3(x_1, x_2)$ as

$$\mathbf{B}_3(x_1, x_2) = \frac{\langle\zeta\hat{\mathbf{R}}(x_1, x_2, \zeta)\rangle}{|\langle\zeta\hat{\mathbf{R}}(x_1, x_2, \zeta)\rangle|} = \frac{\langle\zeta\hat{\mathbf{R}}(x_1, x_2, \zeta)\rangle}{\frac{h}{12} + h\langle\zeta w_3\rangle}. \tag{53}$$

Because of this, the shear strain measures $2\gamma^*_{\alpha 3}$ from eqn (15) are not zero. Furthermore, it may be shown that

$$\mathbf{N}\cdot\mathbf{B}_3 = 1 + O(\epsilon_e^2). \tag{54}$$

The orientation of the kinematical deformed plate triad $\mathbf{B}_i$ relative to $\mathbf{b}_i$ can be specified in terms of orientation angles, Rodrigues parameters, or any suitable angular displacement parameters for arbitrarily large rotation.

For the second approximation, we will consider the terms in the strain expressions which contribute terms to the strain energy of order $\mu c^2(h/l)^2$. The Jaumann strain for the second approximation becomes

$$\Gamma_e = \varepsilon + \zeta Kh + \partial_e w_{\parallel}, \quad 2\Gamma_s = w'_{\parallel} + 2\gamma^* + \partial_t w_3, \quad \Gamma_t = w'_3, \tag{55}$$

where

$$\partial_t = h \begin{bmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \end{bmatrix}, \quad \partial_e = h \begin{bmatrix} \dfrac{\partial}{\partial x_1} & 0 \\ \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1} \\ 0 & \dfrac{\partial}{\partial x_2} \end{bmatrix}. \tag{56}$$

According to the rules of variational–asymptotical analysis, here we introduce another set of warping functions:

$$w_\parallel = v_\parallel, \quad w_3 = v_3 + D_{\perp 1}\varepsilon + D_{\perp 2}Kh, \tag{57}$$

where $v_\parallel$ and v_3 are warpings for the second approximation. These must then be substituted back into the strain energy. In the extensional strain, the term $\partial_e v_\parallel$ turns out to be of the order $\varepsilon(h/l)^2$, which is of a higher order by $(h/l)^2$ than the other terms in Γ_e. However, in the strain energy this unusual term's contribution is of the same order as all the others which are pertinent to the second approximation, i.e. $\mu\varepsilon^2\ (h/l)^2$. The two components of the strain energy turn out to be

$$2J_\parallel = \langle(\varepsilon+\zeta Kh)^{\mathrm{T}} D_\parallel(\varepsilon+\zeta Kh)\rangle + \underline{2\langle(\varepsilon+\zeta Kh)^{\mathrm{T}} D_\parallel \partial_e v_\parallel\rangle},$$

$$2J_\perp = \langle D_t v_3'^2\rangle + \langle[v_\parallel' + 2\gamma^* + \partial_t(D_{\perp 1}\varepsilon + D_{\perp 2}Kh)]^{\mathrm{T}} D_s[v_\parallel' + 2\gamma^* + \partial_t(D_{\perp 1}\varepsilon + D_{\perp 2}Kh)]\rangle, \tag{58}$$

where the underlined term in the extensional energy is the "unusual" one discussed above. The strain energy is to be minimized with respect to the perturbed warpings $v_\parallel$ and v_3 subject to constraints obtained from eqns (14) and (52) as

$$\langle v_3\rangle = \langle v_\parallel\rangle = \langle\zeta v_\parallel\rangle = 0. \tag{59}$$

With the application of the constraints in eqn (59), the underlined term in $J_\parallel$ will vanish if $D_\parallel$ is constant through the thickness (i.e. the same in each layer of a laminated plate). For example, the material constants which make up $D_\parallel$ are constant through the thickness for homogeneous plates, for laminated plates with the same material in all layers and whose lamina are each angle plies with building units of the form $[\theta, -\theta]$, and for laminated plates whose lamina are each woven cloth. In general, however, this term is quite troublesome; indeed, it prevents one from minimizing the strain energy with respect to $v_\parallel$ (because the energy is not a quadratic form with this term present).

We have chosen to neglect this term, with the knowledge that the resulting theory will be asymptotically exact only for the constant $D_\parallel$ case, and an approximation otherwise. Alternatively, one could develop a higher-order plate theory, with a sufficient number of additional kinematical variables to allow this term to be "killed" with additional constraints. Such a development would be more complex than the one we have pursued here, but would appear to be superior to theories in which the number of kinematical variables is dependent on the number of laminae.

Without the underlined term, it can be seen that $J_\parallel$ remains independent of the warping, as in the first approximation. Thus, it is only necessary henceforth to minimize $J_\perp$. This too can be simplified. The first part of $J_\perp$, which is due to out-of-plane warping, is decoupled from the in-plane warping. The minimum value of this can be reached with the constraints $\langle v_3\rangle = 0$, when $v_3 = 0$. Let us now investigate the remaining part. The remaining functional to be minimized is

$$2J_\perp = \langle[v_\parallel' + 2\gamma^* + \partial_t(D_{\perp 1}\varepsilon + D_{\perp 2}Kh)]^{\mathrm{T}} D_s[v_\parallel' + 2\gamma^* + \partial_t(D_{\perp 1}\varepsilon + D_{\perp 2}Kh)]\rangle \tag{60}$$

with constraints taken from the second and third of eqns (59). Let us then define

$$C'_{\perp 1} = -D_{\perp 1}, \quad C'_{\perp 2} = -D_{\perp 2} \tag{61}$$

and make a change of variable

$$v_\parallel = -2\gamma^*\zeta + \partial_t(C_{\perp 1}\varepsilon + C_{\perp 2}Kh) + 2\gamma\zeta + \overline{v_\parallel}. \tag{62}$$

One can then express $J_\perp$ simply as

$$2J_\perp = \langle(\overline{v_\parallel}' + 2\gamma)^T D_s(\overline{v_\parallel} + 2\gamma)\rangle. \tag{63}$$

Applying the constraints on $v_\parallel$ in eqns (59) to eqn (62), one can obtain

$$\langle\overline{v_\parallel}\rangle = \langle\zeta\overline{v_\parallel}\rangle = 0 \tag{64}$$

if

$$2\gamma = 2\gamma^* - 12\partial_t(E_{\perp 1}\varepsilon + E_{\perp 2}Kh) \tag{65}$$

and

$$\langle C_{\perp 1}\rangle = \langle C_{\perp 2}\rangle = 0, \tag{66}$$

where

$$E_{\perp 1} = \langle C_{\perp 1}\zeta\rangle, \quad E_{\perp 2} = \langle C_{\perp 2}\zeta\rangle. \tag{67}$$

Here eqn (66) is employed in order to find the integration constants coming from eqn (61). Here it should once more be noted that in performing the integrations with respect to ζ, inter-lamina continuity on the warping $v_\parallel$ and on $C_{\perp\alpha}$ must be satisfied. If these conditions are met, all integration constants can be determined uniquely.

We now need to solve the transformed variational statement with the functional $J_\perp(\overline{v_\parallel})$ and constraints given by eqn (64). Following the usual steps of the variational calculus one can obtain the minimum as

$$\overline{v_\parallel} = \mathscr{C}2\gamma, \tag{68}$$

where

$$\mathscr{C} = \frac{1}{12}(D_{s0} - 8D_{s2})(E_0 - 8E_2)^{-1} - I_2\zeta, \quad D'_{s0} = D_s^{-1}, \quad D'_{s2} = \frac{\zeta^2}{2}D_s^{-1},$$
$$\langle D_{s0}\rangle = 0, \quad \langle D_{s2}\rangle = 0, \quad E_0 = \langle D_{s0}\zeta\rangle, \quad E_2 = \langle D_{s2}\zeta\rangle, \tag{69}$$

and where I_2 is the 2×2 identity matrix. Substitution of $\overline{v_\parallel}$ from eqns (68) and (69) and 2γ from eqn (65) into eqn (62), one can find the in-plane warping as

$$v_\parallel = \partial_t(C_{\perp 1} - 12\zeta E_{\perp 1})\varepsilon + \partial_t(C_{\perp 2} - 12\zeta E_{\perp 2})Kh + \mathscr{C}2\gamma. \tag{70}$$

Due to in-plane warping, it is evident that the second approximation thus allows the line elements which are originally normal to the undeformed plate to deform so that they are no longer straight and, due to shear deformation, no longer normal to the deformed plate. Substitution of this result into the original functional $J_\perp$ gives the minimum value of this functional, that is the shear energy

$$J_{\perp} = \tfrac{1}{2} 2\gamma^{\mathrm{T}} \mathcal{G} 2\gamma, \tag{71}$$

where

$$\mathcal{G} = \langle \mathcal{C}^{\mathrm{T}} D_s \mathcal{C} \rangle, \quad \mathcal{C} = \tfrac{1}{12} D_s^{-1} (1 - 4\zeta^2)(E_0 - 8E_2)^{-1}. \tag{72}$$

This completes the solution for the second approximation. Recall that this theory is asymptotically correct if $D_{\parallel}$ is constant through the thickness; thus its accuracy will be dependent on the type of construction. It should be noted that $\mathcal{G}$ is a function only of D_s which, in turn, is a function of the material shear moduli, G_{13} and G_{23}, and the ply angle for each lamina. If the material is the same in every lamina and if $G_{13} = G_{23}$, the transverse shear stiffness reduces to

$$\mathcal{G} = \tfrac{5}{6} D_s \quad \text{(same material each lamina and } G_{13} = G_{23}\text{)}, \tag{73}$$

similar to the transverse shear stiffness for plates made of isotropic materials as first derived by Reissner (1985). Otherwise one must determine $\mathcal{G}$ from eqns (72).

Further simplifications in the warping and transverse shear strain measure can be made for a homogeneous, monoclinic plate. Equations (70) and (65) reduce to

$$v_{\parallel} = \frac{1}{2}\left(\zeta^2 - \frac{1}{12}\right)\partial_t D_{\perp}\varepsilon + \frac{1}{6}\zeta\left(\zeta^2 - \frac{3}{20}\right)\partial_t D_{\perp} K h - \frac{5}{3}\zeta\left(\zeta^2 - \frac{3}{20}\right) 2\gamma \quad \text{(homogeneous)},$$

$$2\gamma = 2\gamma^* + \frac{1}{60}\partial_t D_{\perp} K h, \tag{74}$$

while eqn (73) applies as is to this case.

The characteristics of the second approximation may be summarized as: (1) normal line elements of the undeformed plate do not remain straight and normal to the deformed plate average surface; (2) the transverse shear stresses are not zero; (3) the transverse normal strain is *not zero*; and (4) the transverse normal stress is zero.

NONLINEAR PLATE ANALYSIS

The expression for the strain energy per unit area of a laminated plate (excluding boundary-layer phenomena) is now available from eqns (50) and (71) as

$$hJ = \frac{1}{2}\begin{Bmatrix} \varepsilon \\ K \\ 2\gamma \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} A & B & 0 \\ B^{\mathrm{T}} & D & 0 \\ 0 & 0 & \mathcal{G} \end{bmatrix} \begin{Bmatrix} \varepsilon \\ K \\ 2\gamma \end{Bmatrix}. \tag{75}$$

This expression for the strain energy is quite simple in form, in spite of the fact that the theory is valid for *large displacements and large rotations* (which enter through nonlinear expressions for the generalized strains).

Now, we can give the final form of the constitutive law. In addition to the force and moment stress resultants given by eqn (49), here we define the transverse shear stress resultant as

$$Q = \langle Z_s \rangle h = h\left[\frac{\partial J}{\partial(2\gamma)}\right]^{\mathrm{T}} = \mathcal{G} 2\gamma. \tag{76}$$

The two-dimensional elastic law follows as

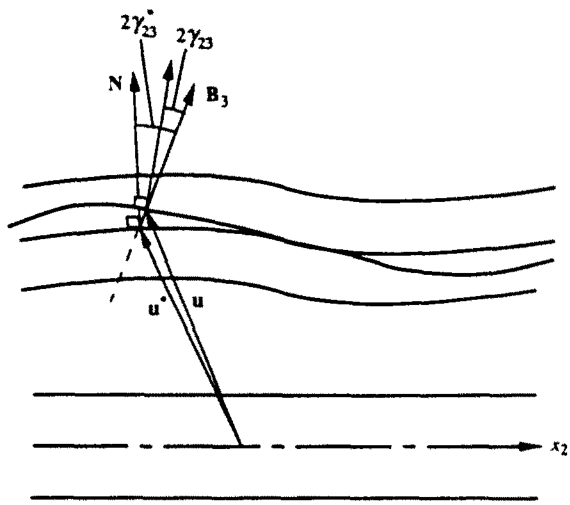

Fig. 2. Planar view of displacement shift to relate u to 2γ.

$$\begin{Bmatrix} N \\ M \\ Q \end{Bmatrix} = \begin{bmatrix} A & B & 0 \\ B^T & D & 0 \\ 0 & 0 & \mathcal{G} \end{bmatrix} \begin{Bmatrix} \varepsilon \\ K \\ 2\gamma \end{Bmatrix}. \tag{77}$$

Now, let us recapitulate the ingredients of the theory as it now stands. The plate boundary value problem is based on five nonlinear intrinsic equilibrium equations, as shown by Berdichevsky (1979) and Hodges *et al.* (1992), which contain the eight stress resultants (N, M and Q) and the eight generalized strain measures (ε, K and 2γ). The stress resultants and generalized strains are related through the eight scalar equations in the elastic law, eqn (77). Recalling the kinematical development above, one can find relations between the generalized strain measures ε, K and $2\gamma^*$ and the average surface displacement measures of $\mathbf{u}^*$ and two rotational parameters needed to specify the direction of $\mathbf{B}_3$.† Finally eqn (65) relates 2γ to $2\gamma^*$.

While we have a complete system of equations, a simpler set would be desirable. One way to accomplish this is to introduce a transformed displacement $\mathbf{u}$ which can be related to 2γ in the same way the $\mathbf{u}^*$ is related to $2\gamma^*$. This can be accomplished by requiring, analogously to the third of eqn (15), that

$$\mathbf{R}_{,\alpha}\cdot\mathbf{B}_3 = 2\gamma_{\alpha 3}. \tag{78}$$

Indeed, it can be shown that a simple shift of the displacement variable along $\mathbf{B}_3$ such that

$$\mathbf{u} = \mathbf{u}^* - 12h(E_{\perp 1}\varepsilon + E_{\perp 2}Kh)\mathbf{B}_3 \tag{79}$$

will satisfy this requirement, will not change K at all, and will change ε only by terms of order ε_e^2.

A sketch of how the displacement is shifted in accordance with eqn (79) is shown in Fig. 2. (The shift is greatly exaggerated in the figure to show its character.) Notice how the transformed surface may be either above or below the average surface depending on the sign of the curvature. Also, recall that neither the transformed displacement $\mathbf{u}$ nor the average surface displacement $\mathbf{u}^*$ correspond to the displacement of a material point on the undeformed plate mid-surface, which is given in eqn (12).

From the above development, displacement [eqns (47) and (70)], strain [eqn (55)] and stress [eqn (21)] for any arbitrary point are obviously available once the plate problem is solved for the generalized strain measures. In Fig. 3 a chart is given which depicts the process of applying the present method to a laminated plate.

† Note that there are only two rotation parameters necessary because the third rotation can always be chosen so that $\varepsilon_{12} = \varepsilon_{21}$. This point is treated in depth by Hodges *et al.* (1992).

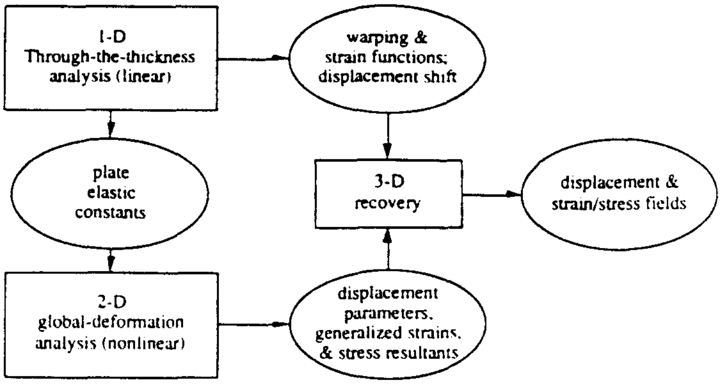

Fig. 3. Overview of plate analysis.

CONCLUDING REMARKS

In this paper we have developed a strain energy function for a laminated, anisotropic plate up through second order in h/l. The present results offer a straightforward approach to solve the complete plate problem, including determination of elastic constants and warping displacement, strain and stress distributions through the thickness. This generality is present even though the number of variables in the present theory is the same as in Reissner–Mindlin theory. The first approximation corresponds to classical laminated plate theory. All quantities through the second approximation, including the shear correction factor, are determined in closed form. Note that the second approximation is asymptotically correct only for plates for which the material constants in the matrix $D_{\parallel}$ do not vary through the thickness. However, it is believed to be adequate for large-deflection engineering analysis of laminated, fiber-reinforced plates each lamina of which possesses monoclinic symmetry about its mid-plane.

A valid criticism of the present work would be that it has ignored edge-zone phenomena. Although these effects can be neglected in classical plate theory (the first approximation), one should take them into account in the higher approximations where the strain energy of a loaded edge may depend on the self-equilibrated part of the load. Here we note the qualitatively different contributions of Berdichevsky (1979) and Arnold and Falk (1989) for consistent development for the edge-zone formulation for homogeneous, isotropic plates. For nonhomogeneous, anisotropic plates, however, this subject is still an open problem.

The present method would need to be extended to treat general nonhomogeneous, anisotropic plates. This would provide a means to treat plates which have fiber-reinforcement through the thickness, such as could result from a three-dimensional braiding manufacturing process. Also, the present method does not provide direct access to the peeling stress in laminated plates. If a more accurate estimation were needed than could be obtained from integration of the equilibrium equations, one would need to consider higher approximations. Finally, provided the present plate theory is adequately tested and shown to perform well relative to other theories, extension of this modeling approach to laminated shells may prove to be feasible.

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